Catalan Numbers

Richard P. Stanley

An OEIS entry

OEIS: *Online Encyclopedia of Integer Sequences* (**Neil Sloane**). See `http://oeis.org`. A database of over 270,000 sequences of integers.

An OEIS entry

OEIS: *Online Encyclopedia of Integer Sequences* (**Neil Sloane**). See `http://oeis.org`. A database of over 270,000 sequences of integers.

A000108: $1, 1, 2, 5, 14, 42, 132, 429, \ldots$

$C_0 = 1,\ C_1 = 2,\ C_2 = 3,\ C_3 = 5,\ C_4 = 14, \ldots$

C_n is a **Catalan number**.

An OEIS entry

OEIS: *Online Encyclopedia of Integer Sequences* (**Neil Sloane**). See `http://oeis.org`. A database of over 270,000 sequences of integers.

A000108: $1, 1, 2, 5, 14, 42, 132, 429, \ldots$

$C_0 = 1,\ C_1 = 2,\ C_2 = 3,\ C_3 = 5,\ C_4 = 14, \ldots$

C_n is a **Catalan number**.

Comments. ... This is probably the longest entry in OEIS, and rightly so.

Catalan monograph

R. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

Catalan monograph

R. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

Includes 214 combinatorial interpretations of C_n and 68 additional problems.

History

Sharabiin Myangat, also known as **Minggatu**, **Ming'antu** (明安图), and **Jing An** (c. 1692–c. 1763): a Mongolian astronomer, mathematician, and topographic scientist who worked at the Qing court in China.

History

Sharabiin Myangat, also known as **Minggatu**, **Ming'antu** (明安图), and **Jing An** (c. 1692–c. 1763): a Mongolian astronomer, mathematician, and topographic scientist who worked at the Qing court in China.

Typical result (1730's):

$$\sin(2\alpha) = 2\sin\alpha - \sum_{n=1}^{\infty} \frac{C_{n-1}}{4^{n-1}} \sin^{2n+1}\alpha$$

History

Sharabiin Myangat, also known as **Minggatu**, **Ming'antu** (明安图), and **Jing An** (c. 1692–c. 1763): a Mongolian astronomer, mathematician, and topographic scientist who worked at the Qing court in China.

Typical result (1730's):

$$\sin(2\alpha) = 2\sin\alpha - \sum_{n=1}^{\infty} \frac{C_{n-1}}{4^{n-1}} \sin^{2n+1}\alpha$$

No combinatorics, no further work in China.

Manuscript of Ming Antu

Manuscript of Ming Antu

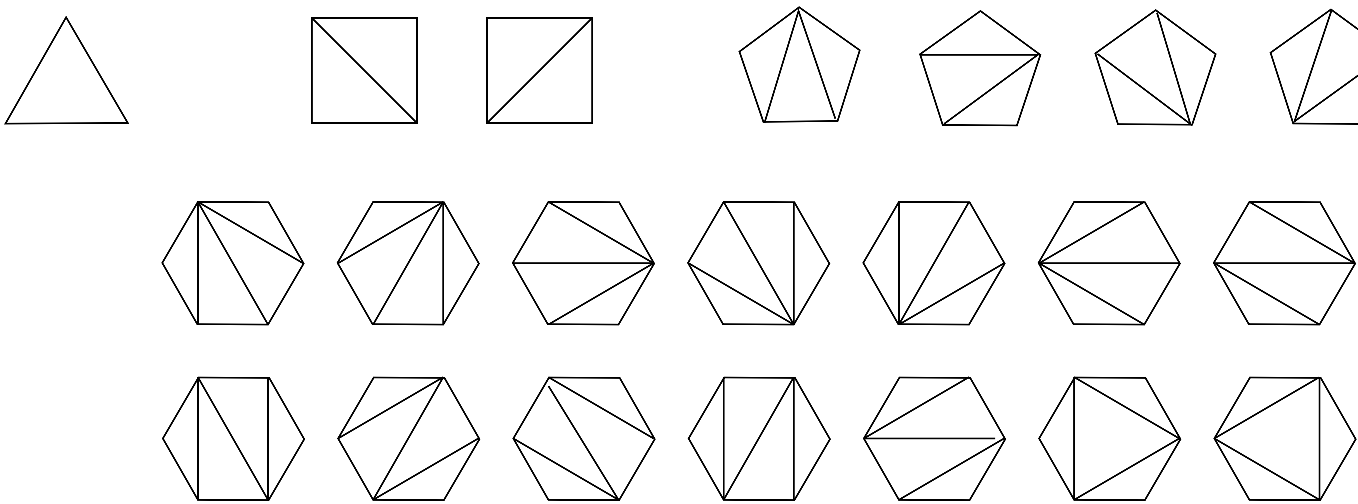
Manuscript of Ming Antu

Manuscript of Ming Antu

少

More history, via Igor Pak

- **Euler** (1751): conjectured formula for number C_n of triangulations of a convex $(n+2)$-gon (**definition** of Catalan numbers). In other words, draw $n-1$ noncrossing diagonals of a convex polygon with $n+2$ sides.



Completion of proof

- **Goldbach and Segner** (1758–1759): helped Euler complete the proof, in pieces.
- **Lamé** (1838): first self-contained, complete proof.

Catalan

- **Eugène Charles Catalan** (1838): wrote C_n in the form $\frac{(2n)!}{n!\,(n+1)!}$ and showed it counted (nonassociative) **bracketings** (or **parenthesizations**) of a string of $n+1$ letters.

Catalan

- **Eugène Charles Catalan** (1838): wrote C_n in the form $\frac{(2n)!}{n!\,(n+1)!}$ and showed it counted (nonassociative) **bracketings** (or **parenthesizations**) of a string of $n+1$ letters.

Born in 1814 in Bruges (now in Belgium, then under Dutch rule). Studied in France and worked in France and Liège, Belgium. Died in Liège in 1894.

Why “Catalan numbers”?

- **John Riordan** (1948): introduced the term “Catalan number” in *Math Reviews*.

Why “Catalan numbers”?

- **John Riordan** (1948): introduced the term “Catalan number” in *Math Reviews*.
- **Riordan** (1964): used the term again in *Math. Reviews*.

Why “Catalan numbers”?

- **John Riordan** (1948): introduced the term “Catalan number” in *Math Reviews*.
- **Riordan** (1964): used the term again in *Math. Reviews*.
- **Riordan** (1968): used the term in his book *Combinatorial Identities*. Finally caught on.

Why “Catalan numbers”?

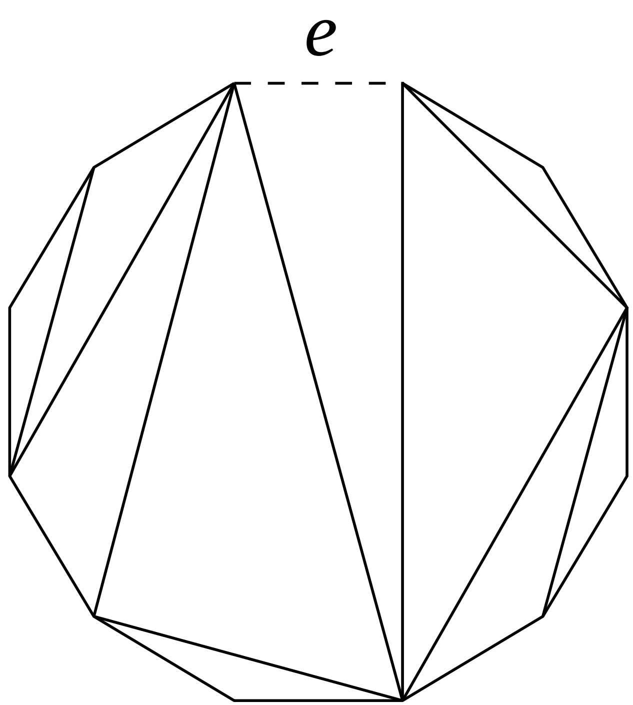
- **John Riordan** (1948): introduced the term “Catalan number” in *Math Reviews*.
- **Riordan** (1964): used the term again in *Math. Reviews*.
- **Riordan** (1968): used the term in his book *Combinatorial Identities*. Finally caught on.
- **Martin Gardner** (1976): used the term in his Mathematical Games column in *Scientific American*. Real popularity began.

The primary recurrence

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1$$

The primary recurrence

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1$$



Solving the recurrence

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1$$

Let $\boldsymbol{y} = \sum_{n \geq 0} C_n x^n$.

Multiply recurrence by x^n and sum on $n \geq 0$.

Solving the recurrence

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1$$

Let $\boldsymbol{y} = \sum_{n\geq 0} C_n x^n$.

Multiply recurrence by x^n and sum on $n \geq 0$.

$$\sum_{n\geq 0} C_{n+1} x^n = \sum_{n\geq 0} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n$$

A quadratic equation

$$\sum_{n\geq 0} C_{n+1}x^n = \sum_{n\geq 0}\left(\sum_{k=0}^{n} C_k C_{n-k}\right)x^n$$

A quadratic equation

$$\sum_{n\geq 0} C_{n+1}x^n = \sum_{n\geq 0}\left(\sum_{k=0}^{n} C_k C_{n-k}\right)x^n$$

Now $x\sum_{n\geq 0} C_{n+1}x^n = \sum_{n\geq 1} C_n x^n = y - 1$.

Moreover, $\sum_{k=0}^{n} C_k C_{n-k}$ is the coefficient of x^n in $\left(\sum_{n\geq 0} C_n x^n\right)^2 = y^2$, since in general, $\sum_{k=0}^{n} a_k b_{n-k}$ is the coefficient of x^n in the product $\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right)$.

A quadratic equation

$$\sum_{n\geq 0} C_{n+1}x^n = \sum_{n\geq 0}\left(\sum_{k=0}^{n} C_k C_{n-k}\right)x^n$$

Now $x\sum_{n\geq 0} C_{n+1}x^n = \sum_{n\geq 1} C_n x^n = y - 1$.

Moreover, $\sum_{k=0}^{n} C_k C_{n-k}$ is the coefficient of x^n in $\left(\sum_{n\geq 0} C_n x^n\right)^2 = y^2$, since in general, $\sum_{k=0}^{n} a_k b_{n-k}$ is the coefficient of x^n in the product $\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right)$.

$$\Rightarrow \frac{y-1}{x} = y^2 \Rightarrow \boldsymbol{xy^2 - y + 1 = 0}$$

Solving the quadratic equation

$$xy^2 - y + 1 = 0 \Rightarrow y = \frac{1 \pm \sqrt{1 - 4x}}{2x}$$

Which sign is correct?

Solving the quadratic equation

$$xy^2 - y + 1 = 0 \Rightarrow y = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Which sign is correct?

Well, in general (Taylor series)

$$(1+u)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} u^n = \sum_{n\geq 0} \alpha(\alpha-1)\cdots(\alpha-n+1)\frac{u^n}{n!}.$$

Solving the quadratic equation

$$xy^2 - y + 1 = 0 \Rightarrow y = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Which sign is correct?

Well, in general (Taylor series)

$$(1+u)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} u^n = \sum_{n \geq 0} \alpha(\alpha-1)\cdots(\alpha-n+1)\frac{u^n}{n!}.$$

Let $u = -4x$, $\alpha = \frac{1}{2}$, to get

$$\sqrt{1-4x} = 1 - 2x - 2x^2 + \cdots .$$

Which sign?

Recall $y = \sum_{n \geq 0} C_n x^n = \frac{1 \pm \sqrt{1-4x}}{2x}$.

Which sign?

Recall $y = \sum_{n\geq 0} C_n x^n = \frac{1\pm\sqrt{1-4x}}{2x}$.

The plus sign gives

$$\frac{1+(1-2x-2x^2+\cdots)}{2x} = \frac{1}{x} - 1 - x + \cdots,$$

which makes no sense. The minus sign gives

$$\frac{1-(1-2x-2x^2+\cdots)}{2x} = 1 + x + \cdots,$$

which is correct.

A formula for C_n

We get

$$\begin{aligned} y &= \frac{1}{2x}(1-\sqrt{1-4x}) \\ &= \frac{1}{2x}\left(1-\sum_{n\geq 0}\binom{1/2}{n}(-4x)^n\right), \end{aligned}$$

where $\binom{1/2}{n} = \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-3}{2})}{n!}$.

A formula for C_n

We get

$$y = \frac{1}{2x}(1 - \sqrt{1-4x})$$
$$= \frac{1}{2x}\left(1 - \sum_{n\geq 0}\binom{1/2}{n}(-4x)^n\right),$$

where $\binom{1/2}{n} = \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-3}{2})}{n!}$.

Simplifies to $y = \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n$, so

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$$

Other combinatorial interpretations

$$\begin{aligned} \mathcal{P}_n &:= \{\text{triangulations of convex } (n+2)\text{-gon}\} \\ \Rightarrow \#\mathcal{P}_n &= C_n \text{ (where } \#S = \text{number of elements of } S) \end{aligned}$$

We want other combinatorial interpretations of C_n, i.e., other sets $\mathcal{S}_n$ for which $C_n = \#\mathcal{S}_n$.

Other combinatorial interpretations

$$\boldsymbol{\mathcal{P}_n} := \{\text{triangulations of convex } (n+2)\text{-gon}\}$$
$$\Rightarrow \#\mathcal{P}_n = C_n \text{ (where } \boldsymbol{\#S} = \text{number of elements of } S\text{)}$$

We want other combinatorial interpretations of C_n, i.e., other sets $\mathcal{S}_n$ for which $C_n = \#\mathcal{S}_n$.

bijective proof: show that $C_n = \#\mathcal{S}_n$ by giving a bijection

$$\varphi\colon \mathcal{T}_n \to \mathcal{S}_n$$

(or $\mathcal{S}_n \to \mathcal{T}_n$), where we already know $\#\mathcal{T}_n = C_n$.

Bijection

Reminder: a **bijection** $\varphi \colon S \to T$ is a function that is one-to-one and onto, that is, for every $t \in T$ there is a unique $s \in S$ for which $\varphi(s) = t$.

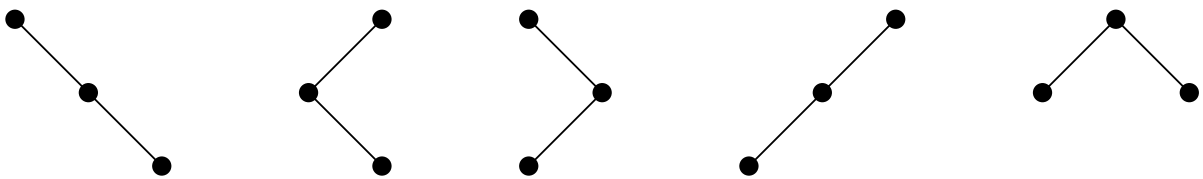
Bijection

Reminder: a **bijection** $\varphi\colon S \to T$ is a function that is one-to-one and onto, that is, for every $t \in T$ there is a unique $s \in S$ for which $\varphi(s) = t$.

If S, T are finite and $\varphi\colon S \to T$ is a bijection, then $\#S = \#T$ (the “best” way to prove $\#S = \#T$).

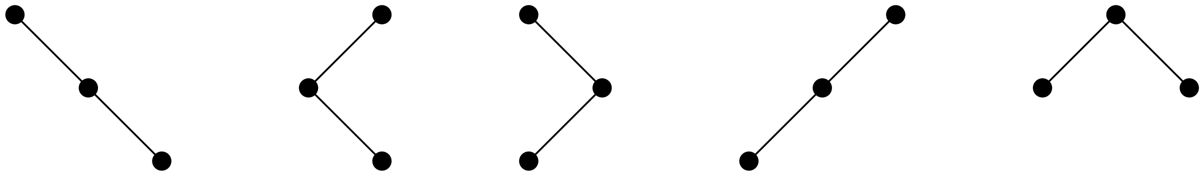
Binary trees

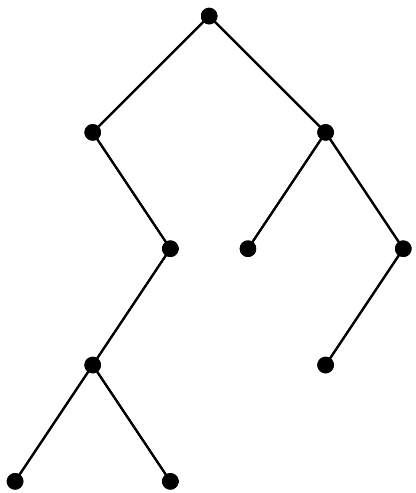
4. Binary trees with n vertices (each vertex has a left subtree and a right subtree, which may be empty)



Binary trees

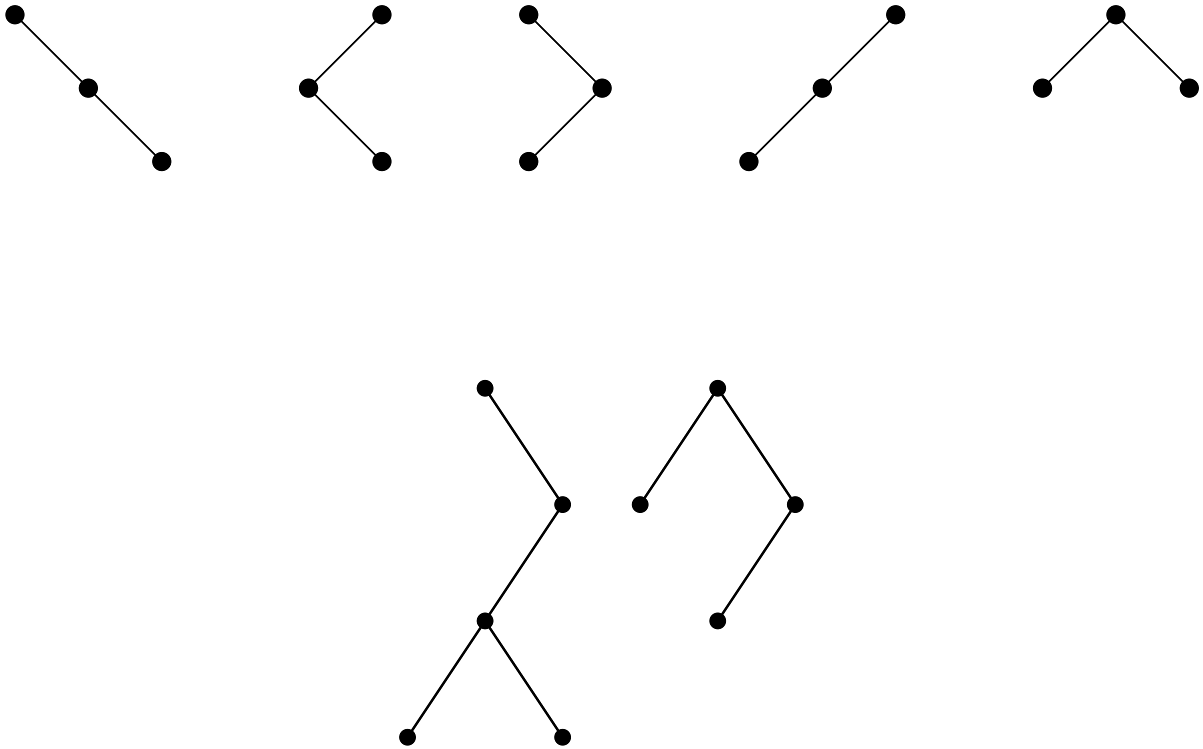
4. Binary trees with n vertices (each vertex has a left subtree and a right subtree, which may be empty)



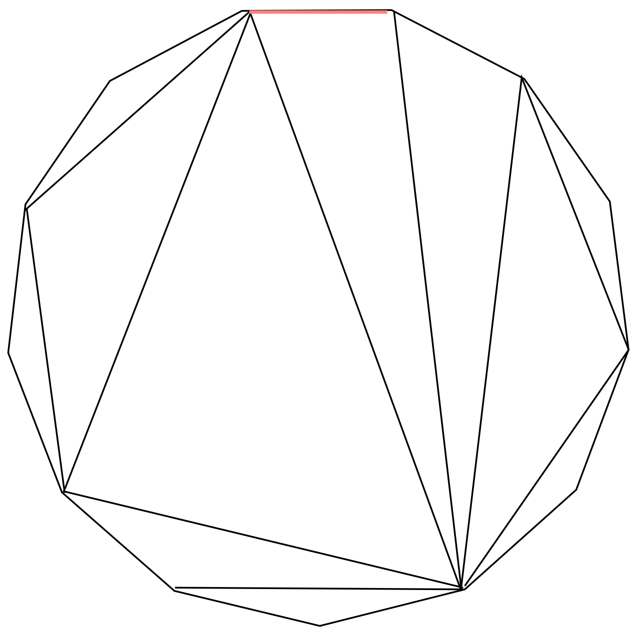


Binary trees

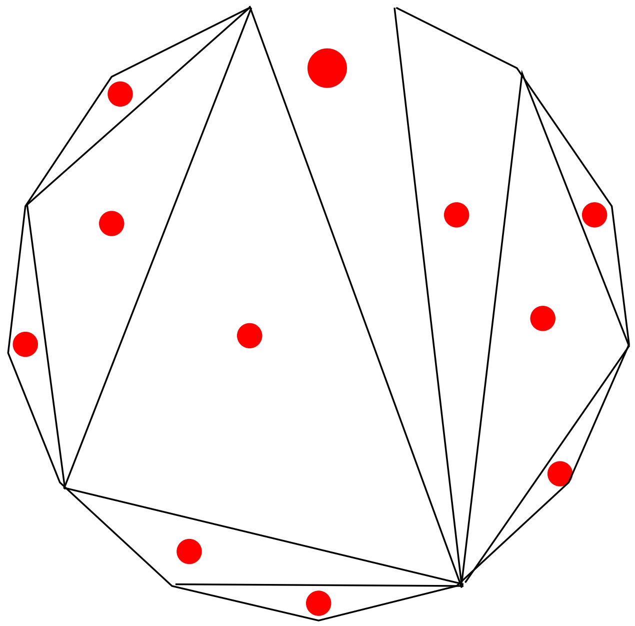
4. Binary trees with n vertices (each vertex has a left subtree and a right subtree, which may be empty)



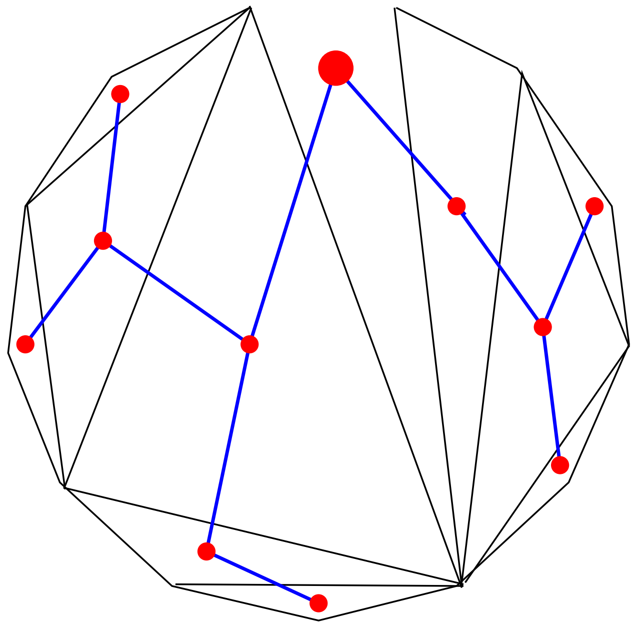
Bijection with triangulations



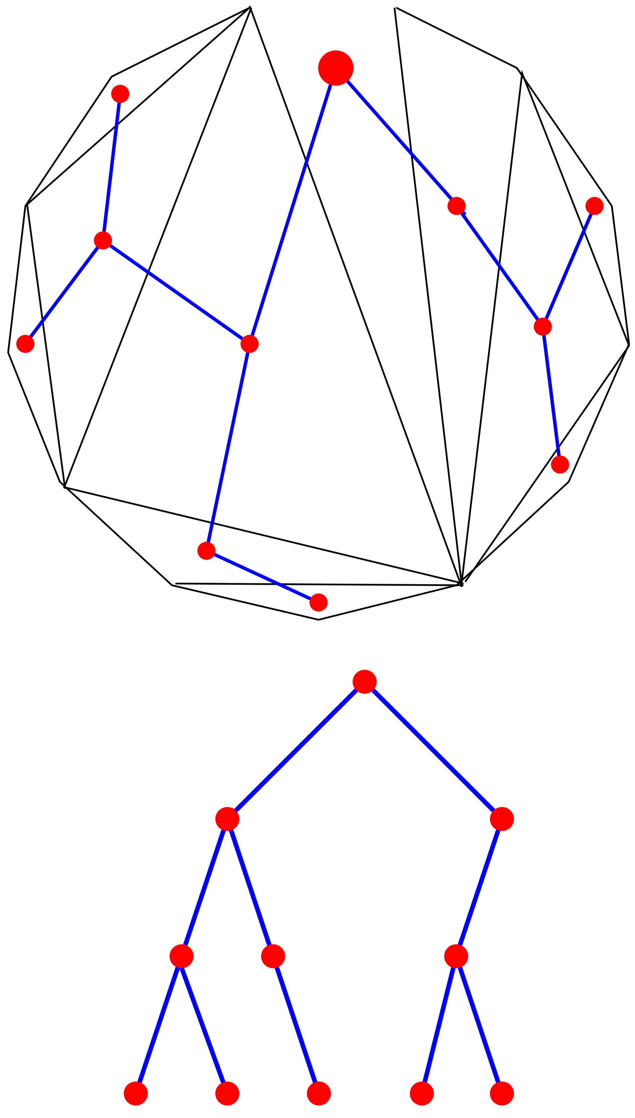
Bijection with triangulations



Bijection with triangulations



Bijection with triangulations



Binary parenthesizations

3. Binary **parenthesizations** or **bracketings** of a string of $n+1$ letters

$$(xx\cdot x)x \quad x(xx\cdot x) \quad (x\cdot xx)x \quad x(x\cdot xx) \quad xx\cdot xx$$

Binary parenthesizations

3. Binary **parenthesizations** or **bracketings** of a string of $n+1$ letters

$$(xx \cdot x)x \quad x(xx \cdot x) \quad (x \cdot xx)x \quad x(x \cdot xx) \quad xx \cdot xx$$

$$((x(xx))x)(x((xx)(xx)))$$

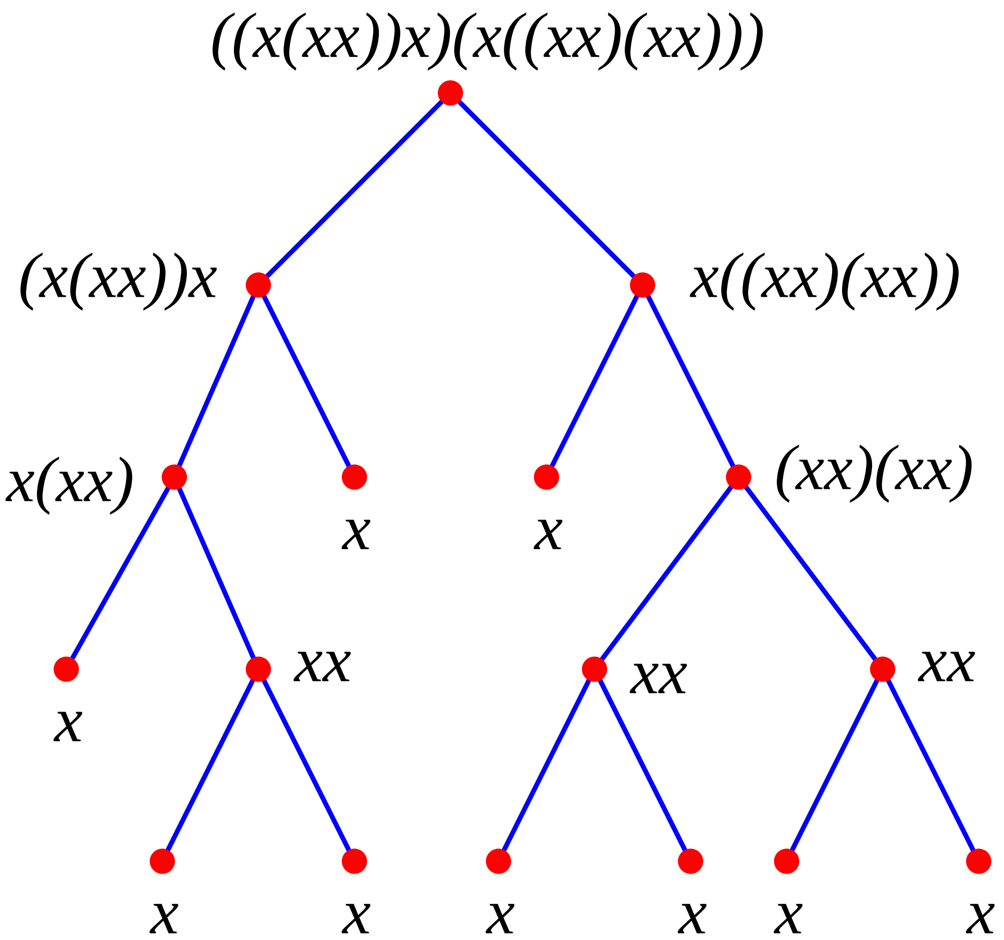
Binary parenthesizations

3. Binary **parenthesizations** or **bracketings** of a string of $n+1$ letters

$$(xx \cdot x)x \quad x(xx \cdot x) \quad (x \cdot xx)x \quad x(x \cdot xx) \quad xx \cdot xx$$

$$((x(xx))x)(x((xx)(xx)))$$

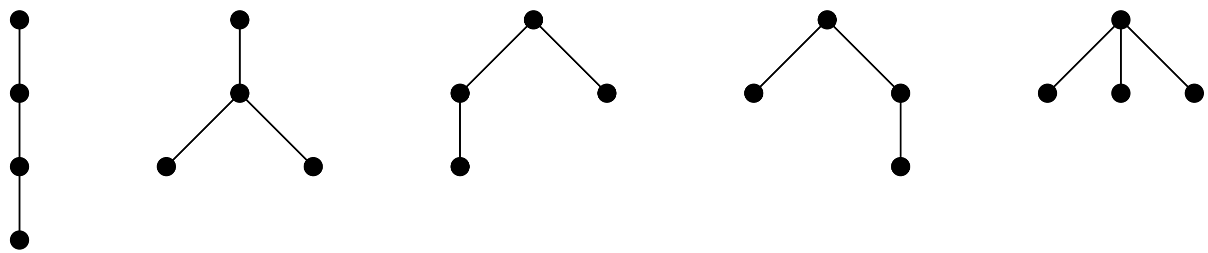
Bijection with binary trees



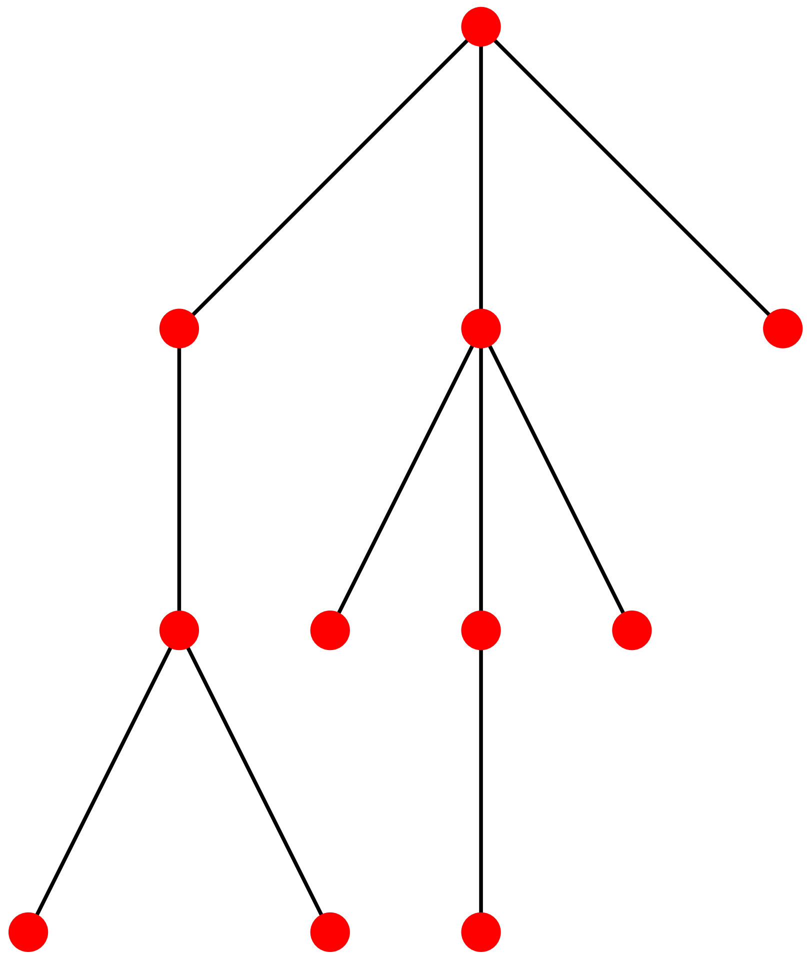
Plane trees

Plane tree: subtrees of a vertex are linearly ordered

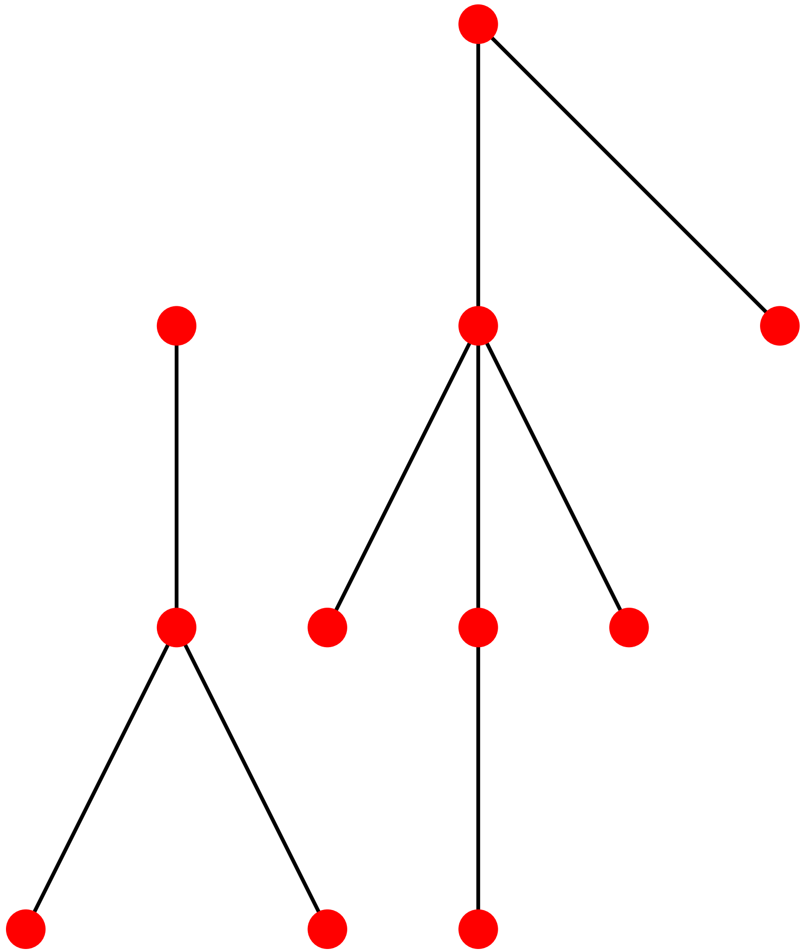
6. Plane trees with $n+1$ vertices



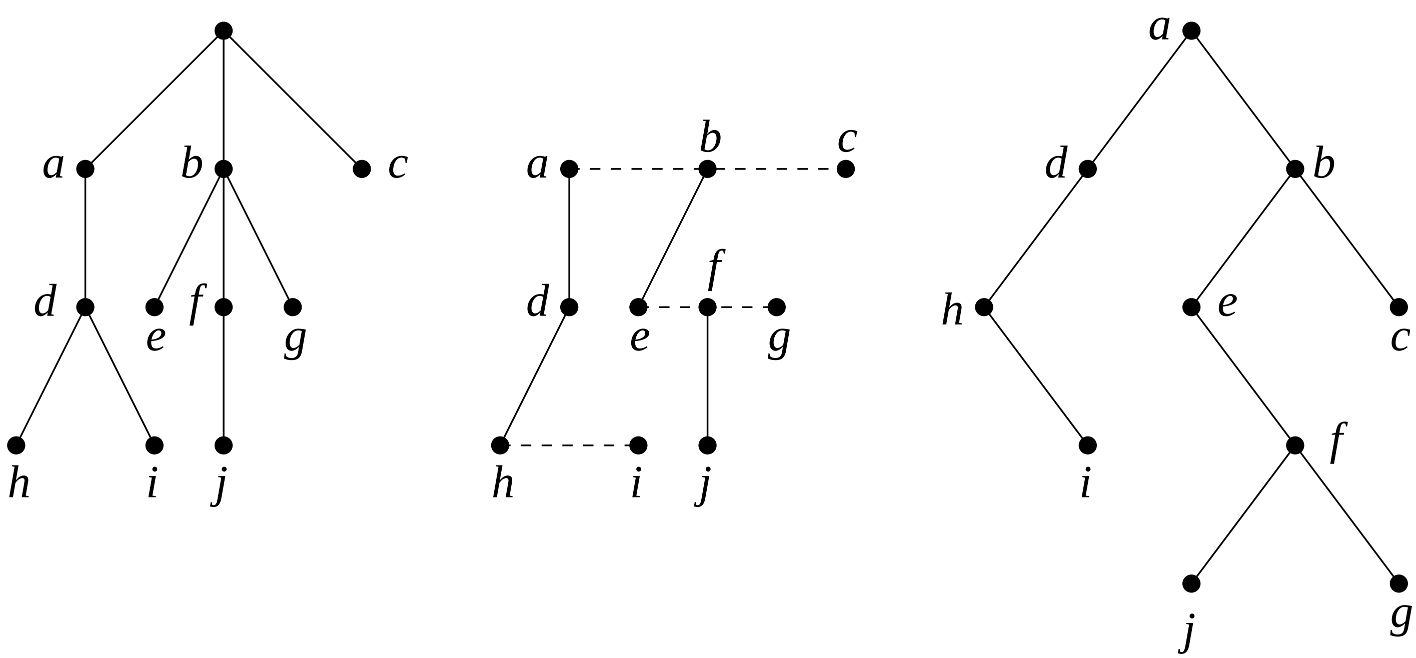
Plane tree recurrence



Plane tree recurrence



Bijection with binary trees



The ballot problem

Bertrand's ballot problem: first published by **W. A. Whitworth** in 1878 but named after **Joseph Louis François Bertrand** who rediscovered it in 1887 (one of the first results in probability theory).

The ballot problem

Bertrand's ballot problem: first published by **W. A. Whitworth** in 1878 but named after **Joseph Louis François Bertrand** who rediscovered it in 1887 (one of the first results in probability theory).

Special case: there are two candidates A and B in an election. Each receives n votes. What is the probability that A will never trail B during the count of votes?

Example. $AABABBBAAB$ is bad, since after seven votes, A receives 3 while B receives 4.

Definition of ballot sequence

Encode a vote for A by 1, and a vote for B by -1 (abbreviated $-$). Clearly a sequence $a_1 a_2 \cdots a_{2n}$ of n each of 1 and -1 is allowed if and only if $\sum_{i=1}^{k} a_i \geq 0$ for all $1 \leq k \leq 2n$. Such a sequence is called a **ballot sequence**.

Ballot sequences

77. Ballot sequences, i.e., sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below)

$$111{-}{-}{-} \quad 11{-}1{-}{-} \quad 11{-}{-}1{-} \quad 1{-}11{-}{-} \quad 1{-}1{-}1{-}$$

Ballot sequences

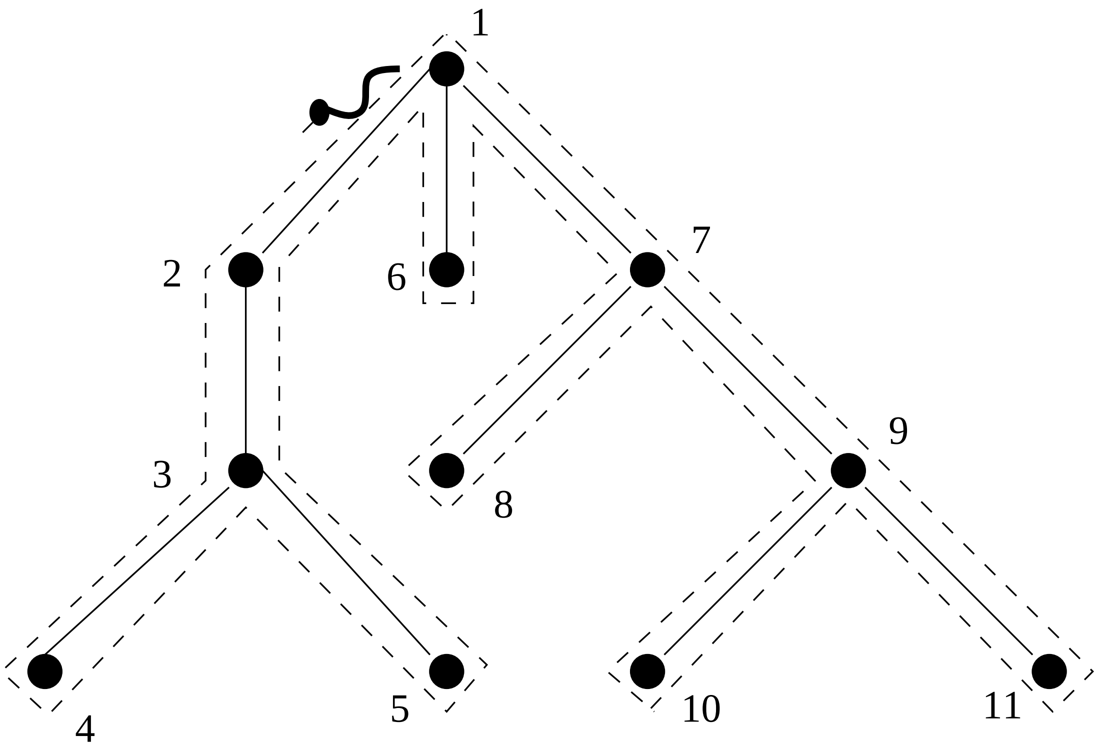
77. Ballot sequences, i.e., sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below)

$$111{-}{-}{-} \quad 11{-}1{-}{-} \quad 11{-}{-}1{-} \quad 1{-}11{-}{-} \quad 1{-}1{-}1{-}$$

Note. Answer to original problem (probability that a sequence of n each of 1's and -1's is a ballot sequence) is therefore

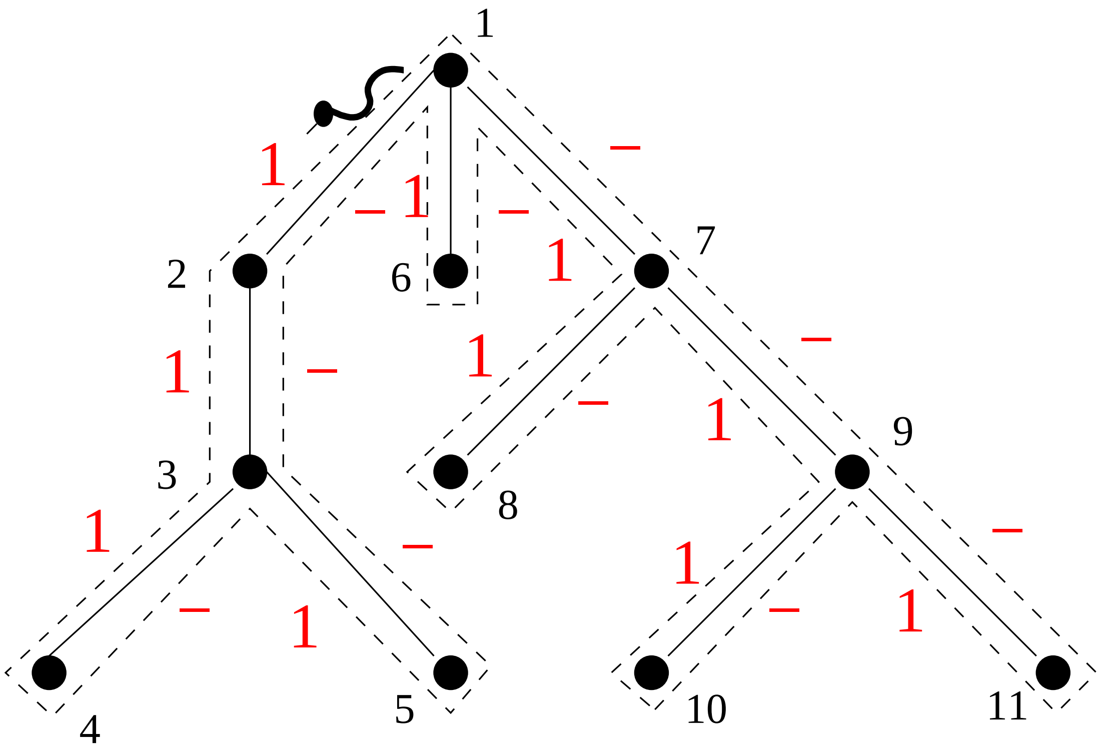
$$\frac{C_n}{\binom{2n}{n}} = \frac{\frac{1}{n+1}\binom{2n}{n}}{\binom{2n}{n}} = \frac{1}{n+1}.$$

Bijection with plane trees



depth first order or **preorder**

Bijection with plane trees



down an edge: $+1$,   up an edge: -1

$1\,1\,1\,-\,1\,-\,-\,-\,1\,-\,1\,1\,-\,1\,1\,-\,1\,-\,-\,-$

Combinatorial proof

Let $\boldsymbol{B_n}$ denote the number of ballot sequences $a_1a_2\cdots a_{2n}$. We will give a direct **combinatorial proof** (no generating functions) that $B_n = \frac{1}{n+1}\binom{2n}{n}$.

Binomial coefficients

Reminder. If $0 \le k \le n$, then $\binom{n}{k}$ is the number of k-element subsets of an n-element set.

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

Example. $\binom{4}{2} = 6$: six 2-element subsets of $\{1,2,3,4\}$ are

$$12 \quad 13 \quad 23 \quad 14 \quad 24 \quad 34.$$

Cyclic shifts

cyclic shift of a sequence $b_0, \ldots, b_m$: any sequence

$$b_i, b_{i+1}, \ldots, b_m, b_0, b_1, \ldots, b_{i-1}, \quad 0 \le i \le m.$$

There are $m+1$ cyclic shifts of $b_0, \ldots, b_m$, but they need not be distinct.

The key lemma

Lemma. *Let* $a_0, a_1, \ldots, a_{2n}$ *be a sequence with* $n+1$ *terms equal to 1 and* n *terms equal to* -1. *All* $2n+1$ *cyclic shifts are distinct since* $n+1$ *and* n *are relatively prime. Exactly one of these cyclic shifts* $a_i, a_{i+1}, \ldots, a_{i-1}$ *has the property that* $a_i = 1$ *and* $a_{i+1}, a_{i+2}, \ldots, a_{i-1}$ *is a ballot sequence.*

Example of key lemma

Let $n = 4$ and consider the sequence $1 - 1\,1 - 1 - - 1$. Five cyclic shifts begin with 1:

$$
\begin{array}{ccccccccccc}
1 & - & 1 & 1 & - & 1 & - & - & 1 & : & \text{no} \\
1 & 1 & - & 1 & - & - & 1 & 1 & - & : & \text{no} \\
1 & - & 1 & - & - & 1 & 1 & - & 1 & : & \text{no} \\
1 & - & - & 1 & 1 & - & 1 & 1 & - & : & \text{no} \\
1 & 1 & - & 1 & 1 & - & 1 & - & - & : & \text{yes!}
\end{array}
$$

Example of key lemma

Let $n = 4$ and consider the sequence $1 - 1 1 - 1 - - 1$. Five cyclic shifts begin with 1:

$$\begin{array}{ccccccccccc} 1 & - & 1 & 1 & - & 1 & - & - & 1 & : & \text{no} \\ 1 & 1 & - & 1 & - & - & 1 & 1 & - & : & \text{no} \\ 1 & - & 1 & - & - & 1 & 1 & - & 1 & : & \text{no} \\ 1 & - & - & 1 & 1 & - & 1 & 1 & - & : & \text{no} \\ 1 & 1 & - & 1 & 1 & - & 1 & - & - & : & \text{yes!} \end{array}$$

Proof of key lemma: straightforward induction argument not given here. $\square$

Enumeration of ballot sequences

The number of sequences $1 = a_0, a_1, \ldots, a_{2n}$ with $n+1$ terms equal to 1 and n terms equal to -1 is $\binom{2n}{n}$. (Choose n of the terms $a_1, \ldots, a_{2n}$ to equal 1.)

Enumeration of ballot sequences

The number of sequences $1 = a_0, a_1, \ldots, a_{2n}$ with $n+1$ terms equal to 1 and n terms equal to -1 is $\binom{2n}{n}$. (Choose n of the terms $a_1, \ldots, a_{2n}$ to equal 1.)

There are $n+1$ cyclic shifts of this sequence that begin with 1. Exactly 1 of them gives a ballot sequence (of length $2n$) when you remove the first term.

Enumeration of ballot sequences

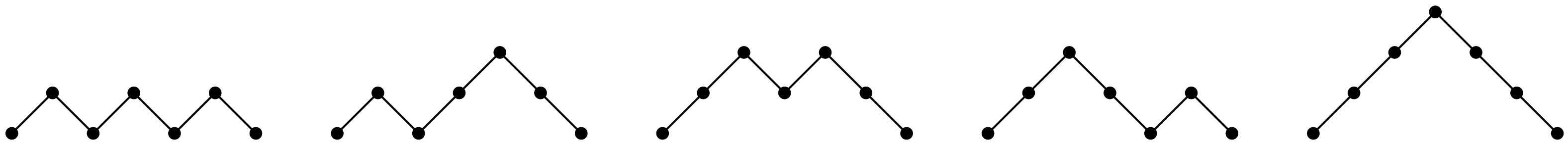
The number of sequences $1 = a_0, a_1, \ldots, a_{2n}$ with $n+1$ terms equal to 1 and n terms equal to -1 is $\binom{2n}{n}$. (Choose n of the terms $a_1, \ldots, a_{2n}$ to equal 1.)

There are $n+1$ cyclic shifts of this sequence that begin with 1. Exactly 1 of them gives a ballot sequence (of length $2n$) when you remove the first term.

Therefore the number of ballot sequences of length $2n$ is $\frac{1}{n+1}\binom{2n}{n} = C_n$.

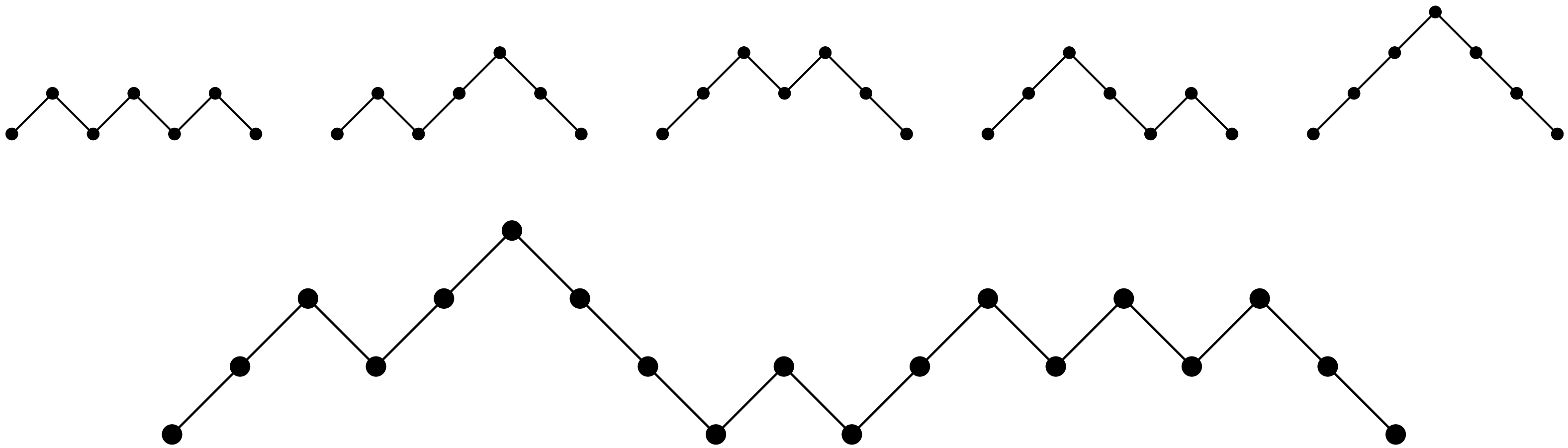
Dyck paths

25. Dyck paths of length $2n$, i.e., lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis



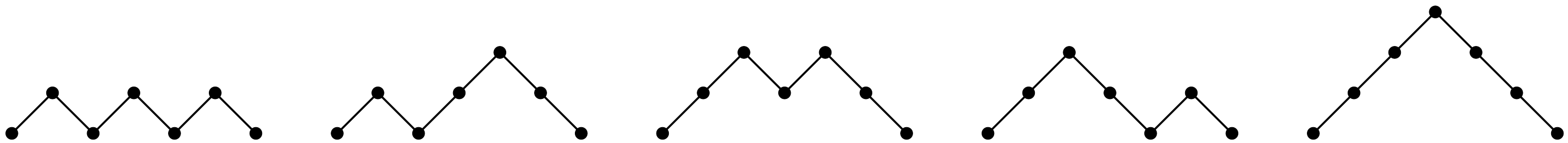
Dyck paths

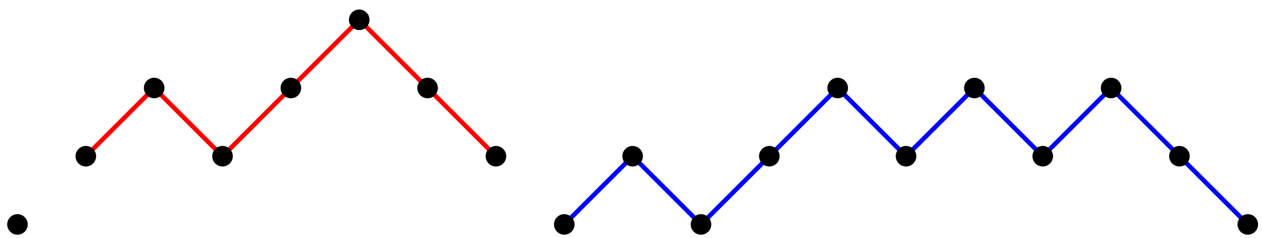
25. Dyck paths of length $2n$, i.e., lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis



Dyck paths

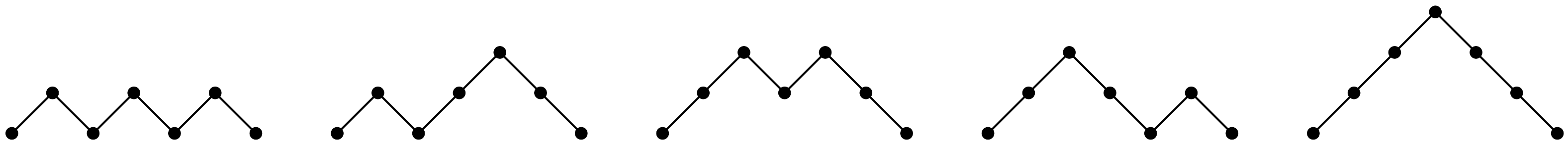
25. Dyck paths of length $2n$, i.e., lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis

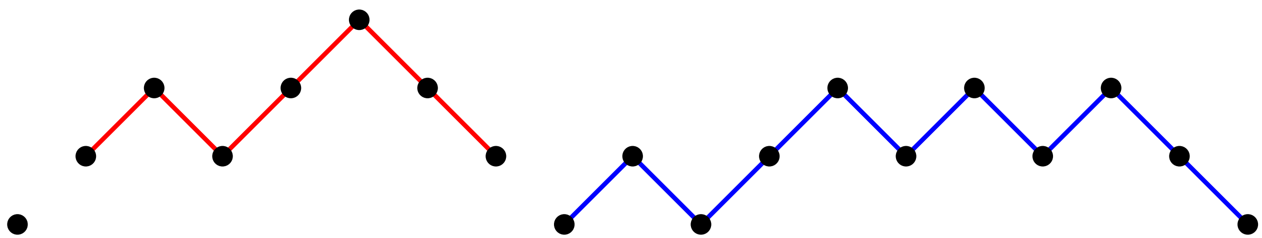




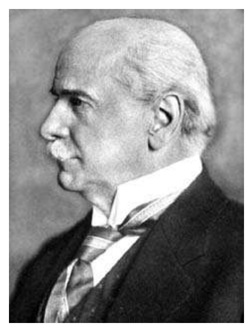
Dyck paths

25. Dyck paths of length $2n$, i.e., lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis

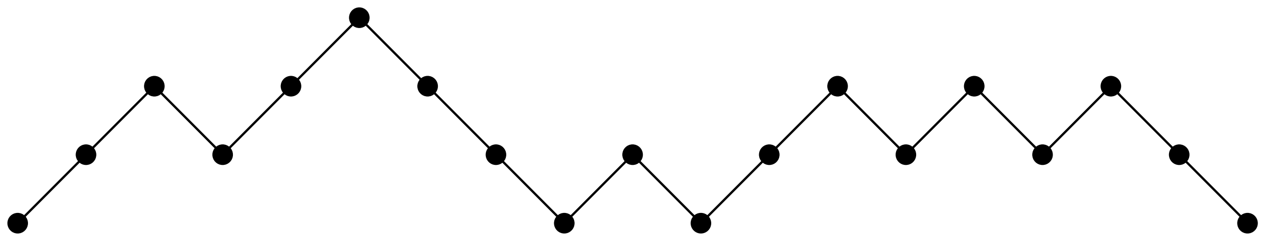




Walther von Dyck (1856–1934)



Bijection with ballot sequences

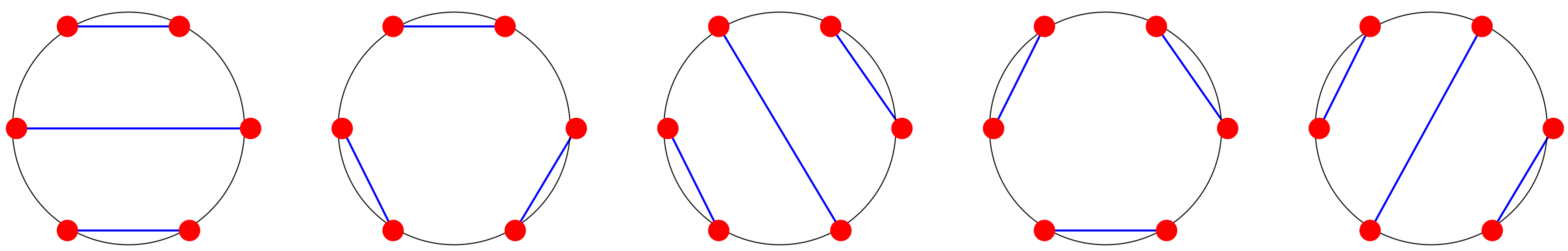


$$1 \; 1 \; - \; 1 \; 1 \; - \; - \; - \; 1 \; - \; 1 \; 1 \; - \; 1 \; - \; 1 \; - \; -$$

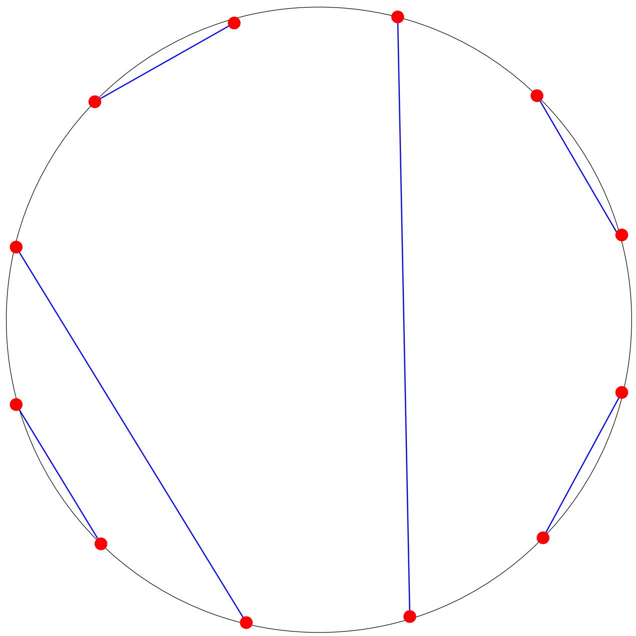
For each upstep, record 1.
For each downstep, record -1.

Noncrossing chords

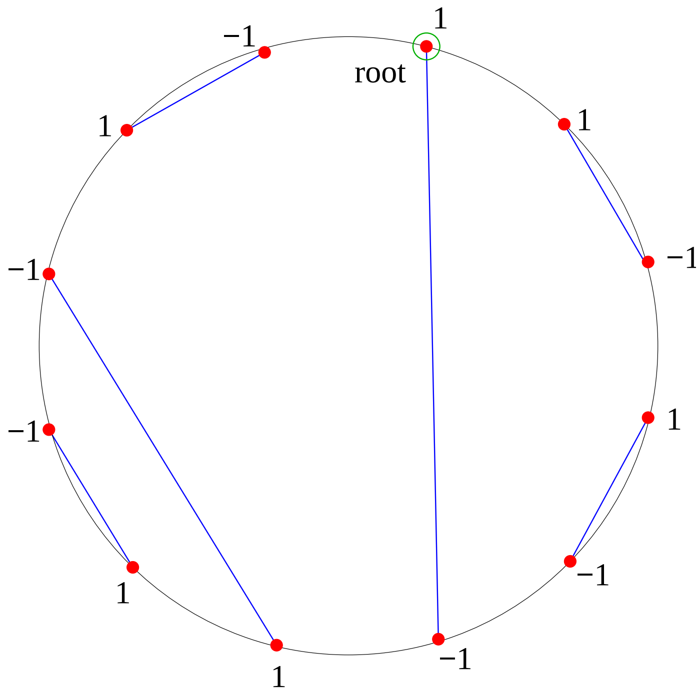
59. n nonintersecting chords joining $2n$ points on the circumference of a circle



Bijection with ballot sequences



Bijection with ballot sequences



$$1\,1\,-\,1\,-\,-\,1\,1\,-\,-\,1\,-$$

312-avoiding permutations

116. Permutations $a_1 a_2 \cdots a_n$ of $1, 2, \ldots, n$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called **312-avoiding**) permutations)

$$123 \quad 132 \quad 213 \quad 231 \quad 321$$

312-avoiding permutations

116. Permutations $a_1 a_2 \cdots a_n$ of $1, 2, \ldots, n$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called **312-avoiding**) permutations)

$$123 \quad 132 \quad 213 \quad 231 \quad 321$$

$$34251768$$

312-avoiding permutations

116. Permutations $a_1 a_2 \cdots a_n$ of $1, 2, \ldots, n$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called **312-avoiding**) permutations)

$$123 \quad 132 \quad 213 \quad 231 \quad 321$$

$$\mathbf{3425\ 768}$$

312-avoiding permutations

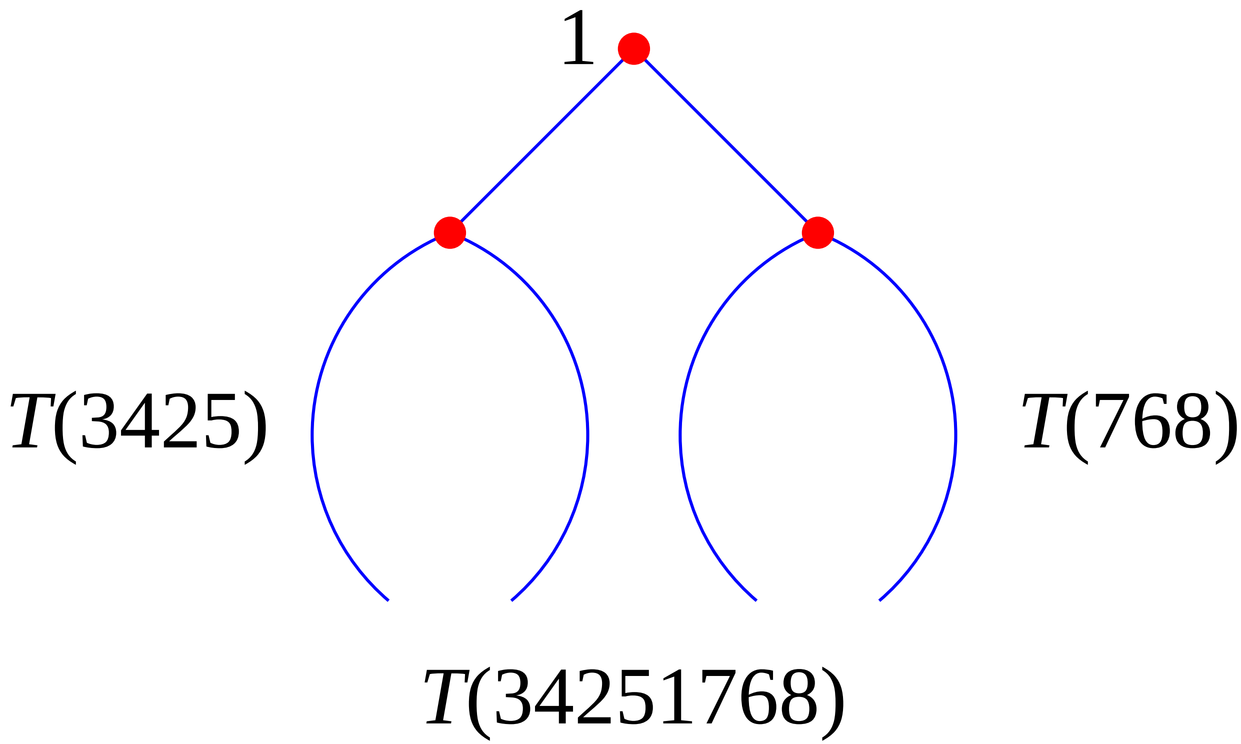
116. Permutations $a_1a_2\cdots a_n$ of $1,2,\ldots,n$ for which there does not exist $i<j<k$ and $a_j<a_k<a_i$ (called **312-avoiding**) permutations)

$$123 \quad 132 \quad 213 \quad 231 \quad 321$$

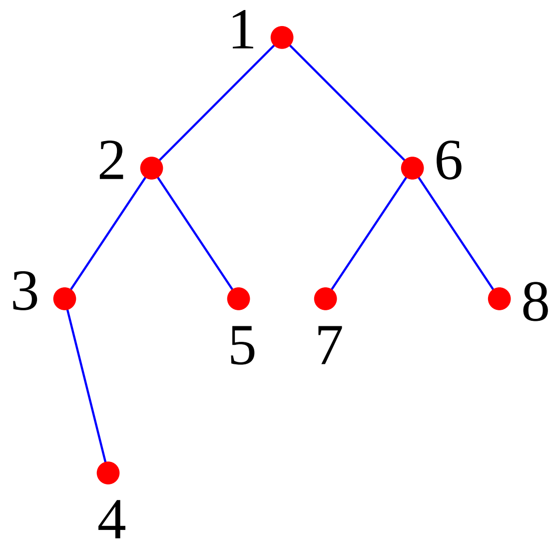
$$\mathbf{3425\ 768}$$

part of the subject of **pattern avoidance**

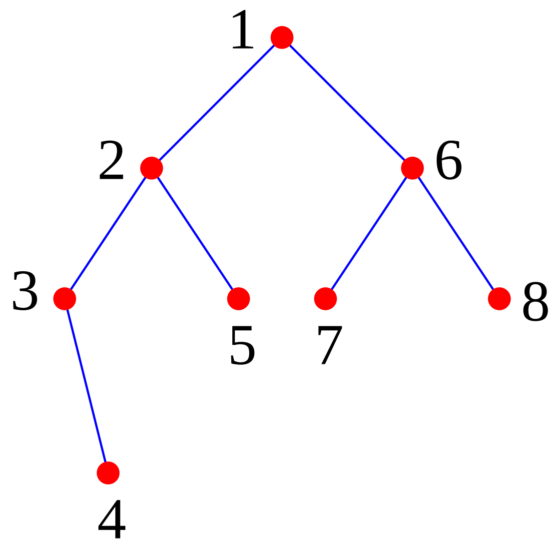
Bijection with binary trees



The tree for 34251768



The tree for 34251768



Note. If we read the vertices in preorder, we obtain 12345678.

Exercise. This gives a bijection between 312-avoiding permutations and binary trees.

321-avoiding permutations

Another example of pattern avoidance:

115. Permutations $a_1 a_2 \cdots a_n$ of $1, 2, \ldots, n$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called **321-avoiding** permutations

$$123 \quad 213 \quad 132 \quad 312 \quad 231$$

321-avoiding permutations

Another example of pattern avoidance:

115. Permutations $a_1 a_2 \cdots a_n$ of $1, 2, \ldots, n$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called **321-avoiding** permutations

$$123 \quad 213 \quad 132 \quad 312 \quad 231$$

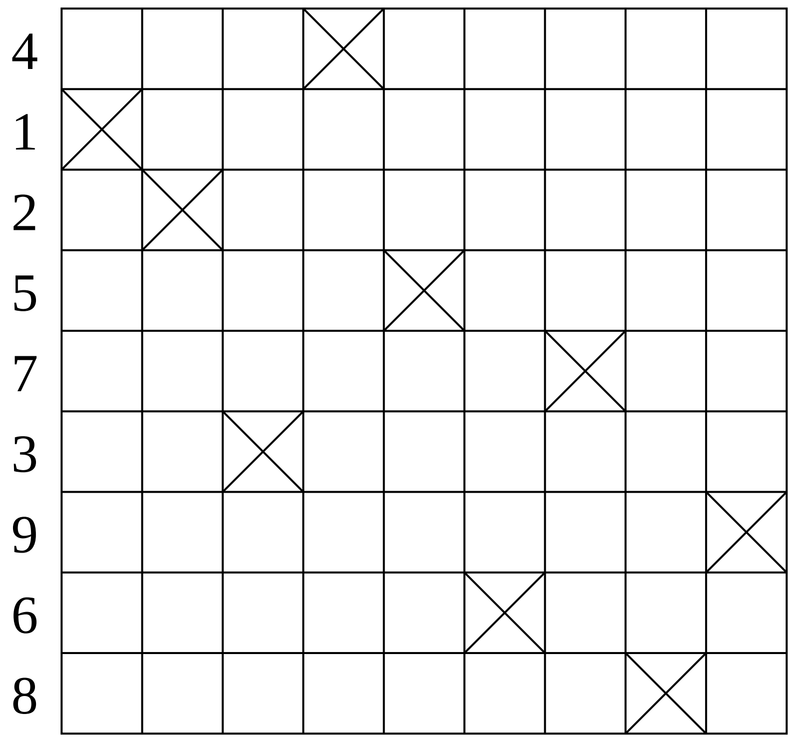
more subtle: no obvious decomposition into two pieces

Bijection with Dyck paths

$$w = 412573968$$

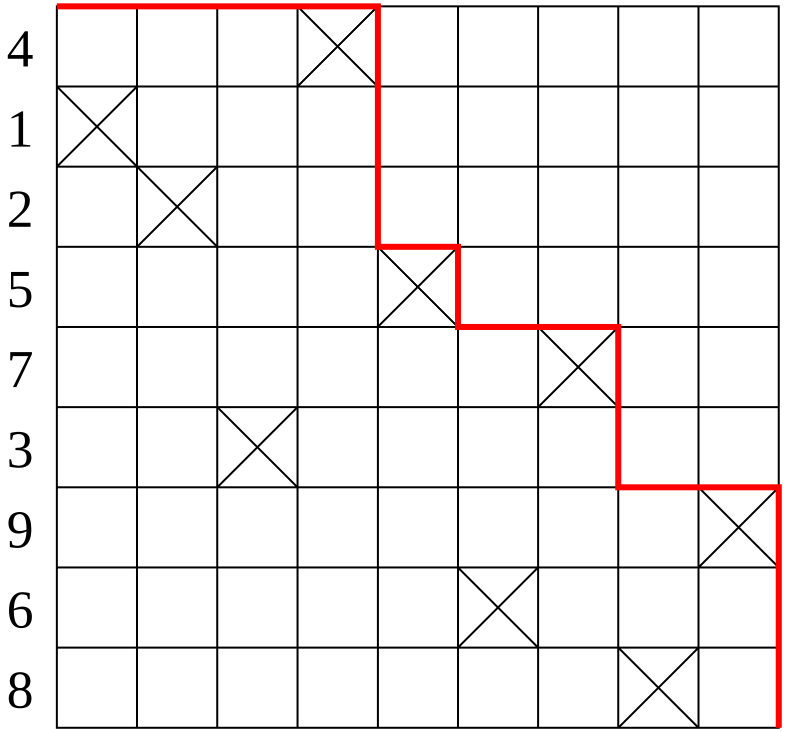
Bijection with Dyck paths

$$w = 412573968$$



Bijection with Dyck paths

$$w = 412573968$$



An unexpected interpretation

92. n-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1a_1a_2\cdots a_n1$, each a_i divides the sum of its two neighbors

$$14321 \quad 13521 \quad 13231 \quad 12531 \quad 12341$$

Bijection with ballot sequences

remove largest, insert bar before the element to its left, then replace bar with 1 and a number with -1, except last two

Bijection with ballot sequences

remove largest, insert bar before the element to its left, then replace bar with 1 and a number with -1, except last two

$$1 \quad 2 \quad 5 \quad 3 \quad 4 \quad 1$$

Bijection with ballot sequences

remove largest, insert bar before the element to its left, then replace bar with 1 and a number with -1, except last two

$$1\ |2\ \ \mathbf{5}\ \ 3\ \ 4\ \ 1$$

Bijection with ballot sequences

remove largest, insert bar before the element to its left, then replace bar with 1 and a number with -1, except last two

$$1\ |2\ \ \mathbf{5}\ |3\ \ \mathbf{4}\ \ 1$$

Bijection with ballot sequences

remove largest, insert bar before the element to its left, then replace bar with 1 and a number with -1, except last two

$$1||2\ \ \mathbf{5}\ |\mathbf{3}\ \ \mathbf{4}\ \ 1$$

Bijection with ballot sequences

remove largest, insert bar before the element to its left, then replace bar with 1 and a number with -1, except last two

$$|1||\mathbf{2}\ \ \mathbf{5}\ |\mathbf{3}\ \ \mathbf{4}\ \ 1$$

Bijection with ballot sequences

remove largest, insert bar before the element to its left, then replace bar with 1 and a number with -1, except last two

$$|1||\mathbf{2}\ \ \mathbf{5}\ |\mathbf{3}\ \ \mathbf{4}\ \ 1$$

$$|1||2\ \ 5\ |3\ \ 4\ \ 1$$

$$\to 1-11--1-$$

Analysis

A65.(b)

$$\sum_{n\geq 0} \frac{1}{C_n} = ??$$

Analysis

A65.(b)

$$\sum_{n\geq 0} \frac{1}{C_n} = ??$$

$$1 + 1 + \frac{1}{2} + \frac{1}{5} = 2.7$$

Analysis

A65.(b)

$$\sum_{n\geq 0} \frac{1}{C_n} = 2 + \frac{4\sqrt{3}\pi}{27} = 2.806\cdots$$

$$1 + 1 + \frac{1}{2} + \frac{1}{5} = 2.7$$

Why?

A65.(a)

$$\sum_{n\geq 0}\frac{x^n}{C_n}=\frac{2(x+8)}{(4-x)^2}+\frac{24\sqrt{x}\sin^{-1}\left(\frac{1}{2}\sqrt{x}\right)}{(4-x)^{5/2}}.$$

Why?

A65.(a)

$$\sum_{n\geq 0}\frac{x^n}{C_n}=\frac{2(x+8)}{(4-x)^2}+\frac{24\sqrt{x}\sin^{-1}\left(\frac{1}{2}\sqrt{x}\right)}{(4-x)^{5/2}}.$$

Sketch of solution. Calculus exercise: let

$$y=2\left(\sin^{-1}\frac{1}{2}\sqrt{x}\right)^2.$$

Then $y=\sum_{n\geq 1}\frac{x^n}{n^2\binom{2n}{n}}$. $\square$

Completion of proof

Recall $y = \sum_{n \geq 1} \frac{x^n}{n^2 \binom{2n}{n}}$. Note that:

Completion of proof

Recall $y = \sum_{n\geq 1} \frac{x^n}{n^2 \binom{2n}{n}}$. Note that:

$$\frac{d}{dx} y = \sum_{n\geq 1} \frac{x^{n-1}}{n\binom{2n}{n}}$$

Completion of proof

Recall $y = \sum_{n\geq 1} \frac{x^n}{n^2\binom{2n}{n}}$. Note that:

$$x\frac{d}{dx}y = \sum_{n\geq 1} \frac{x^n}{n\binom{2n}{n}}$$

Completion of proof

Recall $y = \sum_{n\geq 1} \frac{x^n}{n^2\binom{2n}{n}}$. Note that:

$$\frac{d}{dx} x \frac{d}{dx} y = \sum_{n\geq 1} \frac{x^{n-1}}{\binom{2n}{n}}$$

Completion of proof

Recall $y = \sum_{n\geq 1} \frac{x^n}{n^2\binom{2n}{n}}$. Note that:

$$x^2 \frac{d}{dx} x \frac{dx}{x} y = \sum_{n\geq 1} \frac{x^{n+1}}{\binom{2n}{n}}$$

Completion of proof

Recall $y = \sum_{n\geq 1} \frac{x^n}{n^2\binom{2n}{n}}$. Note that:

$$\frac{d}{dx}x^2\frac{d}{dx}x\frac{dx}{x}y = \sum_{n\geq 1} \frac{(n+1)x^n}{\binom{2n}{n}}$$

Completion of proof

Recall $y = \sum_{n\geq 1} \frac{x^n}{n^2\binom{2n}{n}}$. Note that:

$$\frac{d}{dx}x^2\frac{d}{dx}x\frac{dx}{x}y = \sum_{n\geq 1}\frac{(n+1)x^n}{\binom{2n}{n}}$$

$$= -1 + \sum_{n\geq 0}\frac{x^n}{C_n},$$

etc.

The last slide

The last slide

[redacted]

The last slide