Algorithms and programming
École normale supérieure - Département d'informatique
TD 11:Flow

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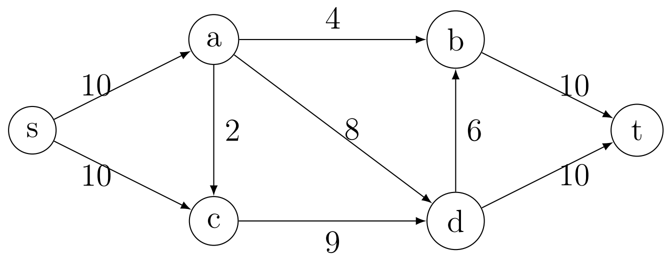
1. Find a maximum flow and a minimum cut by running Ford-Fulkerson algorithm on the following digraph.



2. Two directed edges are said to be *anti-parallel* if they have the same endpoints, but are in opposite directions. Consider a flow network (V, E, s, t, c) and let $e, e' \in E$ be anti-parallel edges. Prove that there exists a maximum flow in which at least one of e, e' has no flow through it.

3. We are given a flow network with several sources and several sinks. Explain how to use an algorithm for finding a maximum flow (in a standard flow network) for this case.

4. Let (V, E, s, t, c) be a network with integral capacities $c(e) \in Z$ for all edges $e \in E$. Prove or refute the following assertions :
   — If all capacities are even then there is a maximal $(s - t)$-flow f such that $f(e)$ is even for all $e \in E$.
   — If all capacities are odd then there is a maximal $(s - t)$-flow f such that $f(e)$ is odd for all $e \in E$.

   needed to be removed so that there would be no path from s to t.

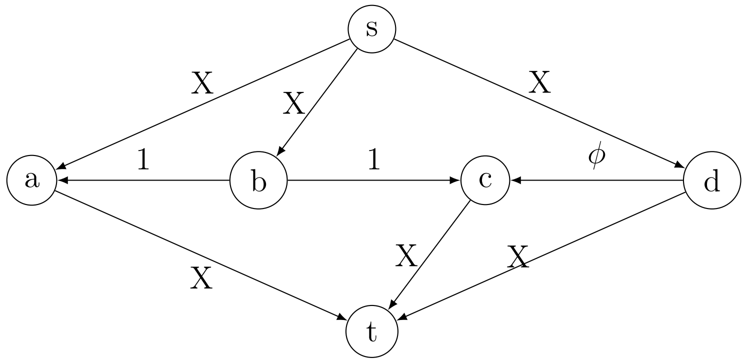
5. *Hall's marriage theorem* : In a bipartite graph with bipartion (A, B), such that $|A| = |B|$, there is a perfect matching if and only if for every $S \subseteq A$, $N(S) \geq |S|$.
   Deduce Hall's marriage theorem from the max-flow min-cut theorem.

6. Construct a network on four vertices for which the Ford-Fulkerson algorithm may need more than a million iterations, depending on the choice of augmenting paths.

7. Suppose you are given a flow network (V, E, s, t, c) with integral capacities. You are also given a maximum flow in it. Now suppose we pick a specific edge $e \in E$ and increase its capacity by one unit. Show how to find a maximum flow in the resulting flow network in time $O(m + n)$

8. Prove that the Ford-Fulkerson algorithm terminates for rational capacities.

9. The goal of this exercise is to show that the Ford-Fulkerson method needs not terminate if we allow irrational edge capacities.
   Consider the following network where X is a large integer and $\phi = \frac{1}{2}(\sqrt{5} - 1)$, and observe that $\phi^n = \phi^{n+1} + \phi^{n+2}$



(a) Show by induction that for any integer $n \geq 0$ the residual capacities of the three horizontal edges can be brought to the values $\phi^n, 0, \phi^{n+1}$.

(b) Conclude that Ford-Fulkerson need not terminate on this network. Does it converge ? Does it converge to a maximum flow ?