Introduction to algorithmics and data structures

L3 Algorithmics and Programming

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Outline

Algorithmics & programming

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- Algorithmics is the study of algorithms: algorithm design, analysis of their performance, etc.

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- An algorithm is the formal specification of the way to solve a given problem: from some input, how to produce the output corresponding to the solution of a problem via elementary operations
- Algorithmics is the study of algorithms: algorithm design, analysis of their performance, etc.
- Programming is the way to turn an algorithm into code in a computer language, so as to execute the algorithm on concrete data

Algorithmics vs programming

- Every algorithm is implementable: it must be described in precise enough terms so that the programming the algorithm is unambiguous
- ... but this does not mean that the program implementing this algorithm is easy to write, as the programmer must take into account machine limits, quirks of the programming language, low-level objects vs high-level concepts, etc.
- Algorithm: abstraction of what is implementable
- The programming language has no influence on what is implementable; all usual programming languages have the same expressive power (Turing-complete)
- ... but a programming language has an impact on ease (cf. `http://pierre.senellart.com/travaux/languages/languages.xml`) or efficiency of implementation

Data structure

- Basic element used in more complex algorithms, reused in various algorithms to solve various problems
- Formal specification of an abstract mathematical object (list, set, function, graph, matrix, etc.), of possible operations on this object (insertion, enumeration, inversion, etc.) and of algorithms realizing them
- Implementable building block, often in the form of a class in object-oriented programming
- Often possible to design different data structures for the same mathematical object, with different efficiency

Outline

Algorithmics & programming

Algorithmic complexity

Elementary data structures

Amortized complexity

References

How to measure the efficiency of an algorithm?

- Attempt at characterizing, from the description of an algorithm, the efficiency of a program that implements it; or, from the description of a problem, the efficiency of a program that implements an algorithm solving the problem
- Different notions of efficiency, different notions of complexity:

  Time complexity computation time of a sequential program
  Space complexity memory space used by a program program
  Communication complexity volume of data exchanged by a distributed system
  Descriptive complexity shortest program length
  Circuit complexity size of the electronic circuit implementing the algorithm

- In this course: first two only (and mostly the first!)

How to compute time complexity

- One assumes every elementary operations appearing in the description of an algorithm:
  - arithmetic operations
  - variable assignments
  - comparisons
  - tests
  - etc.

  takes elementary time, bounded by a constant C
- One sums the number of elementary operations made, as a function of the input size n, e.g., $42 \times n$
- On deduces a bound, here $42 \times n \times C$, on the total time of the algorithm

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- Elementary operation times can be made formal (with the notions of Turing machines, or of Von Neumann machines), but we will skip this

$O()$, $\Omega()$, $\Theta()$ notation

- Let $f : \mathbb{N} \to \mathbb{R}_+$, $g : \mathbb{N} \to \mathbb{R}_+$ be two functions
- One writes $f(n) \in O(g(n))$ (or $f(n) = O(g(n))$) if

$$\exists N \in \mathbb{N}, \exists \alpha \in \mathbb{R}_+^*, \forall n > N \quad f(n) \leqslant \alpha g(n)$$

- One writes $f(n) \in \Omega(g(n))$ (or $f(n) = \Omega(g(n))$) if

$$\exists N \in \mathbb{N}, \exists \alpha \in \mathbb{R}_+^*, \forall n > N \quad f(n) \geqslant \alpha g(n)$$

- One writes $f(n) \in \Theta(g(n))$ (or $f(n) = \Theta(g(n))$) if

$$f(n) = O(g(n)) \quad \text{and} \quad f(n) = \Omega(g(n))$$

Asymptotic time complexity

- One uses the $O()$, $\Omega()$, $\Theta()$ notation and bounds that have been established to indicate the complexity of an algorithm while neglecting C and other constants
- For example, if the elementary operation time τ is bounded by:

$$C_1 \leqslant \tau \leqslant C_2$$

- ... and if on all inputs, algorithm A makes $42 \times n$ operations, then:

$$42 \times C_1 \times n \leqslant \mathrm{T}(\mathcal{A}, n) \leqslant 42 \times C_2 \times n$$

so that $T(\mathcal{A}, n) = \Theta(n)$

Complexity in the worse case, average case

- Usually, one looks for an upper bound on the time of an algorithm, and one looks at the worst case complexity: an upper bound that holds on any input
- Sometimes too restrictive, and then one looks at the average case: on average, for all inputs of a given size, what is a bound on the complexity?
- Assumes that all inputs have the same probability, which is debatable

Simple example: searching an array

Input: Array T with n distinct elements, an element x within x

Output: the position of x in T

```
for i ← 0 to n − 1 do
    if T[i] = x then
        return i
    end if
end for
```

How many elementary operations?

Simple example: searching an array

Input: Array T with n distinct elements, an element x within x
Output: the position of x in T

for $i \leftarrow 0$ to $n - 1$ **do**
    if $T[i] = x$ **then**
        return i
    end if
end for

How many elementary operations?

Worst case

(x in last position)

- n assignments of i
- n comparisons of i with n
- n accesses to $T[i]$
- n comparisons of $T[i]$ with x
- 1 return

$4n + 1$, i.e., $O(n)$

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- n accesses to $T[i]$
- n comparisons of $T[i]$ with x
- 1 return

$4n + 1$, i.e., $O(n)$

Average case

(x in expected position $n/2$)

- $n/2$ assignments of i
- $n/2$ comparisons of i with n
- $n/2$ accesses to $T[i]$
- $n/2$ comparisons of $T[i]$ with x
- 1 return

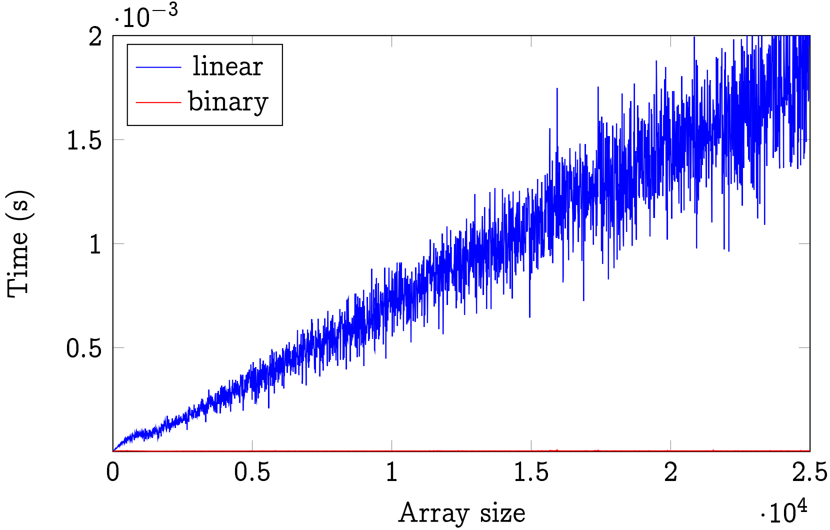
$2n + 1$, i.e., $O(n)$

Asymptotic complexity vs real efficiency

In practice:

- Asymptotic complexity matters. An $O(n)$ algorithm is slower than an $O(\log n)$ one *if n is large enough*
- Sometimes, an algorithm in $O(n^2)$ (i.e., $\leqslant \alpha n^2$) may be more efficient than an algorithm in $O(n)$ (i.e., $\leqslant \beta n$) for common input size n, because $\alpha \ll \beta$
- Sometimes, worst-case complexity is high, but average-case complexity is low and may be the only thing that matters in practice
- The programming language used has a real impact on performance (but, generally, by a constant factor, potentially large)
- Time complexity does not say anything on the potential for parallelization or distribution of an algorithm – other complexity notions are necessary

Linear vs binary search in a sorted array



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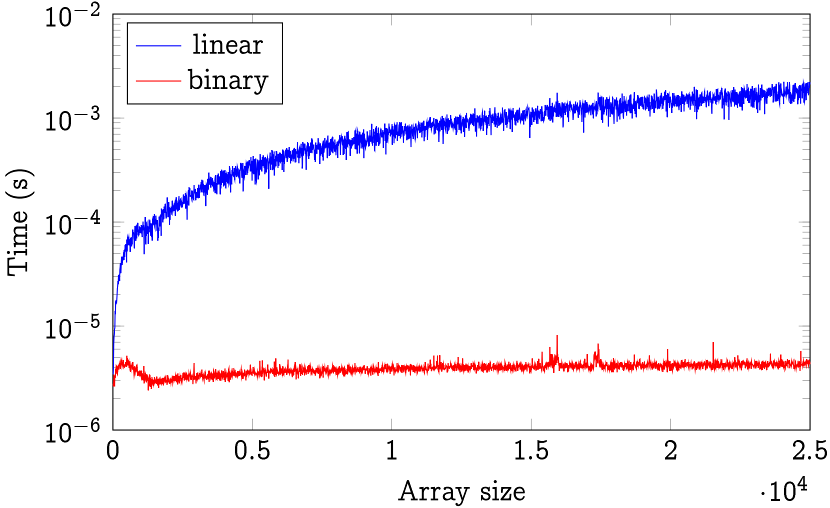
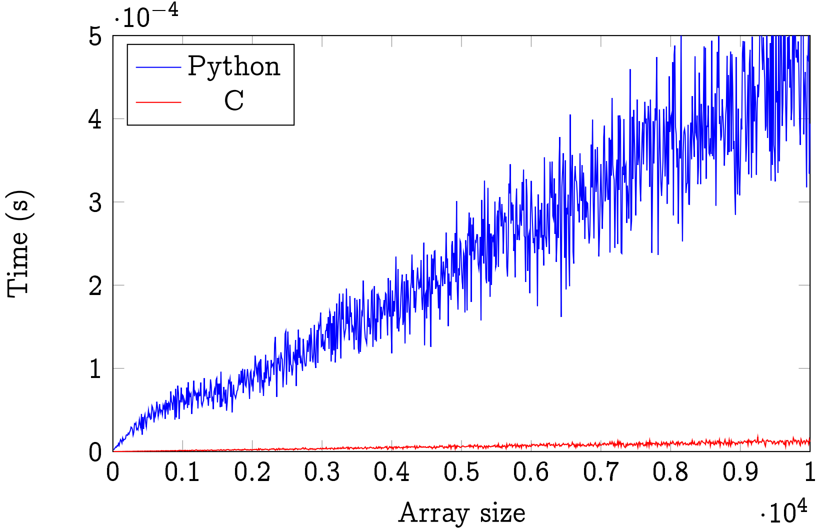


Linear search: Python vs C



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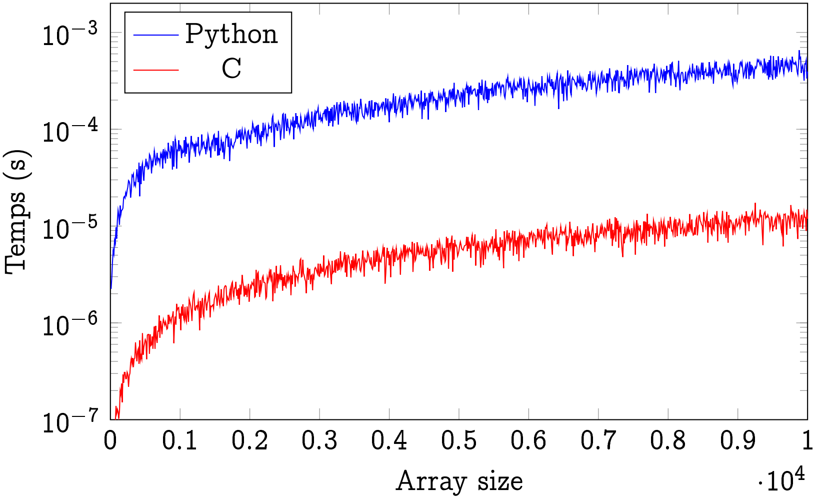


Space complexity

- Same as time complexity, except one counts elementary uses of memory space instead of time of elementary operations
- One often makes simplifying assumptions, such as the fact that any integer fits in constant space
- One does not count the space needed to represent the input
- One also uses $O()$, $\Omega()$, $\Theta()$ as a summary of asymptotic complexity
- For example, array search, space complexity of $O(1)$: just store the variable i in memory (in addition to the inputs T and x), which requires an elementary space, independent of the size of the input

Outline

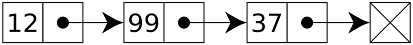
Containers

- Data structures storing a list $L = (l_1, \dots, l_n)$ of elements (e.g., integers, floating-point numbers, complex objects, etc.)
- Diverse specifications of such a structure, allowing different operations, with different efficiency:
  - Random access: given i, access l_i
  - Access at the beginning (l_1), at the end (l_n)
  - Insertion at the beginning (before l_1), at a random position (between l_i and l_{i+1}), at the end (after l_n)
  - Deletion of the first element (l_1), of a random element (l_i), of the last element (l_n)
  - Variant: assuming an ordering on the elements, we suppose $l_1 \leqslant \dots \leqslant l_n$. Access to an ordered element, insertion while following order.

Fixed array

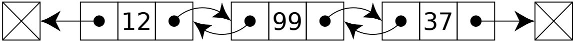
- Contiguous memory area, of a fixed size, pre-allocated
- Random access in $O(1)$
- Insertion, deletion impossible
- As compact as possible, no lost space
- Corresponds to classic arrays of programming languages:
  - bracketed arrays in C or Java
  - `std::array` in C++ 2011
  - `numpy.array` in Python

Linked list



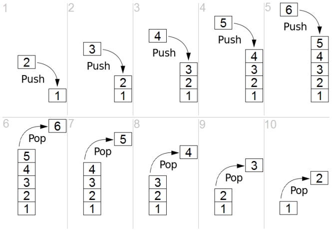
- Each element is stored in a link, which also contains a pointer to the next element
- Also maintain a pointer to the first link
- Access to first element in $O(1)$
- Random access or access to last element in $O(n)$
- Insertion in a random position (once the previous element has been accessed) in $O(1)$
- Deletion of the first element in $O(1)$
- Deletion in $O(1)$ if the previous element is known, in $O(n)$ otherwise
- `std::forward_list` in C++ 2011

Doubly linked list



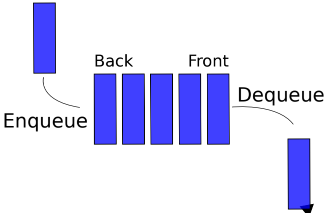
- Each element is stored in a link, which also contains pointers to the previous and next elements
- Also maintain a pointer to the first and last links
- Access to first/last element in $O(1)$
- Random access in $O(n)$
- Insertion in a random position (once the element has been accessed) in $O(1)$
- Deletion of a random element (once it has been accessed) in $O(1)$
- In programming languages:
    - `std::list` in C++
    - `java.util.LinkedList` in Java

Stack (or *LIFO* for *last-in-first-out*)



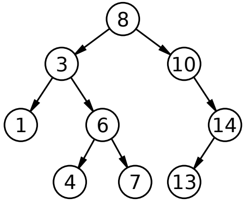
- Abstract data structure, that can be implemented in different ways, e.g., with a singly linked list
- Only possible operations:
  - Access to first element in $O(1)$
  - Insertion at the beginning in $O(1)$
  - Deletion of the first element in $O(1)$
- In programming languages:
  - `std::stack` in C++
  - not explicit in Python, but standard lists can be used
  - `java.util.Stack` in Java

Queue (or *FIFO* for *first-in-first-out*)



- Abstract data structure, that can be implemented in different ways, e.g., with a doubly linked list
- Only possible operations:
  - Access to last element in $O(1)$
  - Insertion at the beginning in $O(1)$
  - Deletion of the last element in $O(1)$
- In programming languages:
  - `std::queue` in C++
  - not explicit in Python, but `collections.deque` can be used
  - `java.util.Queue` interface in Java

Unbalanced binary search tree



- Stores an ordered list of elements
- Binary tree: every node points towards at most two children, a pointer to the root is kept
- For every node storing element l, the subtree rooted at the left child (if it exists) contains elements $\leqslant l$, and the subtree rooted at the right child elements $\geqslant l$
- Access, insertion, deletion of a given element: $O(d)$ where d is the depth of the tree
- No bound on this depth!

Outline

Amortized complexity

- Up to this point, complexity is measured separately for each operation on a data structure
- Sometimes, not possible to bound the time of each individual operation, but possible to bound the average time within a sequence of operations
- Consider a sequence of n operations $o_1, \ldots, o_n$ on a data structure, with cost $c_1, \ldots, c_n$
- Compute the average complexity of an operation o_i, i.e.,

$$\frac{1}{n} \sum_{i=1}^{n} c_i$$

- This is called the amortized complexity
- Only makes sense if a precise definition of the sequence type considered is given

Potential method

- Associate a potential $\Phi(X)$ (real number) to a data structure X
- Consider a sequence of n operations $o_1, \ldots, o_n$, of real costs $c_1, \ldots, c_n$, and corresponding data structures $X_0, \ldots, X_n$
- Define $\hat{c}_i := c_i + \Phi(X_i) - \Phi(X_{i-1})$
- Then: $\frac{1}{n}\sum_{i=1}^{n}\hat{c}_i = \frac{1}{n}\sum_{i=1}^{n} c_i + \frac{1}{n}(\Phi(X_n) - \Phi(X_0))$
- If $\hat{c}_i = O(f(|X|))$, then we also have $\frac{1}{n}\sum_{i=1}^{n} c_i = O(f(|X|))$ if for all n, $\Phi(X_n) \geqslant \Phi(X_0)$
- Often, one takes X_0 the empty data structure and $\Phi(X_0) := 0$, which gives the condition $\Phi(X_n) \geqslant 0$

Application: dynamic array

- Pointer towards a classic array of capacity c + integer $n \leqslant c$ storing the size really used
- Random access in $O(1)$
- Deletion of the last element in $O(1)$ by decreasing n
- Insertion at the end in amortized O(1) complexity (see further)
- Called vector, array, array list in programming languages:
  - `std::vector` in C++
  - lists and array in standard Python, or `numpy.ndarray`
  - `java.util.Vector` and `java.util.ArrayList` in Java

Insertion at the end of a dynamic array

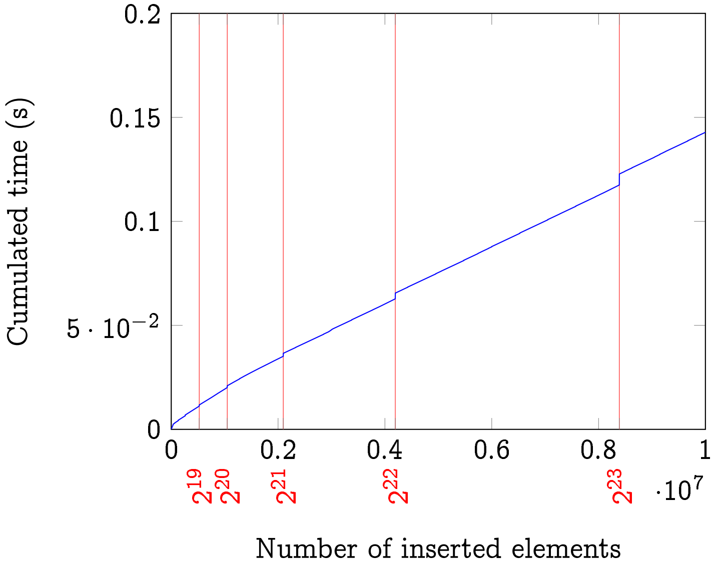
Input: array T of capacity c and size n, element x
Output: array T of size $n+1$ with $T[n] = x$

if $n = c$ **then**
  allocate new array T' of size $\max(2 \times c, 1)$
  for $i \leftarrow 0$ to $n-1$ **do**
    $T'[i] \leftarrow T[i]$
  end for
  deallocate array pointed by T
  make T point to T'
  $c \leftarrow \max(2 \times c, 1)$
end if
$T[n] \leftarrow x$
$n \leftarrow n + 1$

Analysis of amortized complexity

Blackboard analysis for a sequence of n insertions, discussion of the case of arbitrary sequences.

Insertion in a std::vector in C++



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Algorithmic complexity

Elementary data structures

Amortized complexity

References

References

- Generalities on algorithmics: Chap. 1 of [Cormen et al., 2009, 2010]
- Basics of complexity analysis of an algorithm: Chap. 2 and 3 of [Cormen et al., 2009, 2010]
- Elementary data structures: Chap. 10 of [Cormen et al., 2009, 2010]
- Amortized complexity: Chap. 17 of [Cormen et al., 2009, 2010]

Bibliography

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Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Algorithmique*. Dunod, 3rd edition, 2010. ISBN 978-2-100-54526-1. URL https://www.dunod.com/sciences-techniques/algorithmique-cours-avec-957-exercices-et-158-problemes.

Used resources

The image of a queue is due to Vegpuff (Wikimedia), CC-BY-SA-3.0.